

Fourier series

1D-1

— or how to find
what frequencies
(or ———)
are present in a signal.

2.7 — Representation by an
orthogonal signal set
(basic concept)

2.8 Trigonometric Fourier Series
(Anything is the sum of sinusoids)

2.9 Exponential Fourier Series
(Anything is the sum of exponentials)

2.10 Numerical Computation of coefficients
(How a computer can do it. Covered in "digital signal processing")

HW —

2.1-1, 2, 5

2.3-1, 2, 3, 4

2.5-2, 5

2.6-1

2.8-2

Signal representation by orthogonal signal set

1D-2

Given any signal,

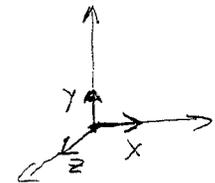
We can represent it as the sum of
orthogonal basis components.

First, look at vectors (2.7.1)

Consider a 3-dimensional space,

with 3 mutually orthogonal basis vectors

x, y, z



— we have a coordinate system —

the basis vectors are of unit length
in the direction of an axis.

Consider a vector from the origin
to a point $(3, 5, 2)$.

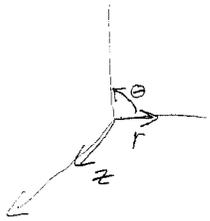
We can represent it as $g = 3x + 5y + 2z$

where x, y, z are orthogonal basis vectors.

1D-3

The x, y, z form is common

Actually we could use any set of mutually orthogonal vectors.



$$r, \theta, z$$

where $r = \sqrt{x^2 + y^2}$
 $\theta = \arctan\left(\frac{y}{x}\right)$

$$g = 5.83r + 59.04\theta + 2z$$

Change of basis is an important concept in signal processing.

(Color TV)

Video: R, G, B
 Transmitted: Y, I, Q

If the set of basis vectors is not complete, there will be some error -

$$g \approx c_1x + c_2y$$

$$e = g - (c_1x + c_2y)$$

↑ error

(Missing z component)

1D-4

Completeness means that it is impossible

to find another vector (x_4)

that is orthogonal to all that we have. (x, y, z)

A set of vectors (X_i) is mutually orthogonal

if:

$$X_m \cdot X_n = \begin{cases} 0 & m \neq n \\ |X_n|^2 & m = n \end{cases}$$

If the basis set is complete — g can be expressed as the sum:

$$g = \sum c_i X_i$$

and coefficients are:

$$c_i = \frac{g \cdot X_i}{X_i \cdot X_i} \quad i = 1, 2, 3, \dots$$

Orthogonal signal space

ID-5

Instead of using a sum (vectors)
use an integral --
inner product instead of dot product, etc.

Backup --- define Orthogonality in complex signals:
(2.5.3)

Recall:

$$\text{Energy: } E_x = \int_{t_1}^{t_2} |x(t)|^2 dt$$

$$\text{Error: } e(t) = g(t) - c x(t)$$

$$E_e = \int_{t_1}^{t_2} |g(t) - c x(t)|^2 dt$$

Math-identity:

$$\begin{aligned} |u+v|^2 &= (u+v)(u^* + v^*) \\ &= |u|^2 + |v|^2 + u^*v + uv^* \end{aligned}$$

conjugate

Substituting + solving --

$$c = \frac{1}{E_x} \int_{t_1}^{t_2} g(t) x^*(t) dt$$

↑
conjugate

Orthogonality is defined as:

$$\int_{t_1}^{t_2} x_1(t) x_2^*(t) dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2} x_1^*(t) x_2(t) dt = 0$$

contrast to $\int_{t_1}^{t_2} x_1(t) x_2(t) dt$ for real signals.

ID-6

back to orthogonal signal space...

We want to represent our signal $g(t)$

as a sum of orthogonal signals $x_1(t), x_2(t), x_3(t), \dots$
over the time interval $[t_1, t_2]$ --

The signals x_n are orthogonal if:

$$\int_{t_1}^{t_2} x_m(t) x_n^*(t) dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

If all energies $E_n = 1$, the set is normalized
and is called an orthonormal set.

We can make an orthonormal set by dividing
all $x_n(t)$ by $\sqrt{E_n}$.

So ----

$$g(t) \approx c_1 x_1(t) + c_2 x_2(t) + \dots$$

$$= \sum_{n=1}^N c_n x_n(t) \quad t_1 \leq t \leq t_2$$

$$c_n = \frac{\int_{t_1}^{t_2} g(t) x_n^*(t) dt}{\int_{t_1}^{t_2} x_n^2(t) dt}$$

$$= \frac{1}{E_n} \int_{t_1}^{t_2} g(t) x_n^*(t) dt \quad n = 1, 2, \dots$$

Remember ---
 $g(t)$ is the
signal

x_n are a set
of orthonormal
basis signals

If the orthogonal set is complete

error energy = 0

and the representation is exact:

$$g(t) = c_1 x_1(t) + c_2 x_2(t) + \dots + c_N x_N(t)$$

↑
how =

(but we rarely get this)

More typically \dots c_N gets smaller as $N \rightarrow \infty$

This is a generalized Fourier series

Specific cases:

Let $x_n =$ sines and cosines

$$= \{1, \cos \omega_0 t, \sin \omega_0 t, \cos 2\omega_0 t, \sin 2\omega_0 t, \cos 3\omega_0 t, \sin 3\omega_0 t, \dots\}$$

→ Trigonometric Fourier Series

Let $x_n =$ exponentials

$$= e^{jn\omega_0 t} \quad n = 0, \pm 1, \pm 2, \dots$$

→ Exponential Fourier series.

Others that we use:

Chebyshev polynomials — Filters

Bessel functions — FM

Walsh functions — data compression

Parseval's theorem —

The energy of a signal equals the sum of the energies in all its components.

$$E_g = c_1^2 E_1 + c_2^2 E_2 + c_3^2 E_3 + \dots$$

$$= \sum_n c_n^2 E_n$$

Energy of a component is:

$$c_n^2 E_n$$

↑ the coefficient ↙ Energy in the basis function

Kirchoff's laws are special cases of Parseval's theorem.

Trigonometric Fourier Series -

ID-9

use sines + cosines ---

$n\omega_0$ is the "nth harmonic" of ω_0
 n is an integer

ω_0 is the fundamental

other terms are harmonics

{ $1, \leftarrow$ 0th harmonic
 $\cos \omega_0 t, \sin \omega_0 t,$ actually --- $\cos(0) = 1$
 $\cos 2\omega_0 t, \sin 2\omega_0 t,$ = DC component,
 \dots }

$$g(t) = a_0 + a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t) + a_2 \cos(2\omega_0 t) + b_2 \sin(2\omega_0 t) + \dots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$t_1 \leq t \leq t_1 + T_0$$

↑
Some time interval.

To determine the coefficients ---

ID-10

$$a_n = \frac{\int_{t_1}^{t_1+T_0} g(t) \cos(n\omega_0 t) dt}{\int_{t_1}^{t_1+T_0} \cos^2(n\omega_0 t) dt} \quad (\text{from p. 6})$$

The denominator is $\frac{T_0}{2}$ for $n=1, \dots$

and T_0 for $n=0$

So it can be simplified --

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} g(t) dt$$

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} g(t) \cos(n\omega_0 t) dt \quad n=1, 2, 3, \dots$$

likewise --

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} g(t) \sin(n\omega_0 t) dt \quad n=1, 2, 3, \dots$$

"Compact" trigonometric Fourier series - 1D-11

A shorter way to write the same thing;

$$a_n \cos(n\omega t) + b_n \sin(n\omega t) = C_n \cos(n\omega t + \theta_n)$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

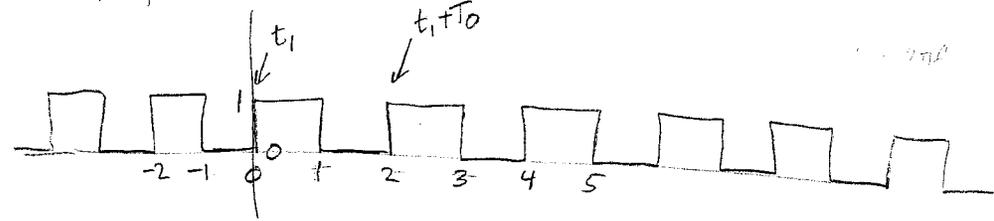
$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

$$C_0 = a_0$$

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega t + \theta_n)$$

Example:

1D-12



$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} g(t) dt$$

$$T_0 = 2$$

$$f_0 = \frac{1}{2}$$

$$\omega = \frac{2\pi}{2} = \pi$$

$$= \frac{1}{2} \left[\int_0^1 1 dt + \int_1^2 0 dt \right]$$

$$= \frac{1}{2}$$

$$a_1 = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} g(t) \cos(\omega t) dt$$

$$= \frac{2}{2} \left[\int_0^1 1 \cos(\pi t) dt + \int_1^2 0 \cos(\pi t) dt \right]$$

$$= \frac{1}{\pi} \sin(\pi t) \Big|_0^1 - \frac{1}{\pi} \sin(\pi t) \Big|_1^2$$

$$= \sin \pi - \sin 0$$

$$= 0$$