

Narrow band FM --

7C-1

Reminder --- it is a nonlinear system.

Superposition does not apply.

For narrow band -- keep modulation small --

$$|K_f a(t)| \ll 1$$

Recall ---

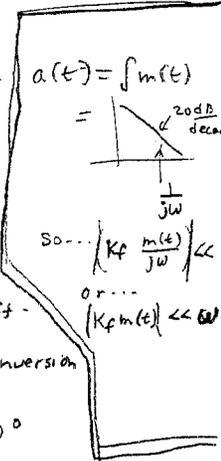
$$\Phi_{FM} = A \cos(\omega_c t + K_f a(t))$$

↑ This is a phase shift --

$$= \pi \text{ means phase inversion}$$

$$= \frac{\pi}{2} = 1.57 \text{ means } 90^\circ$$

so $\ll 1$ means a very small phase shift.

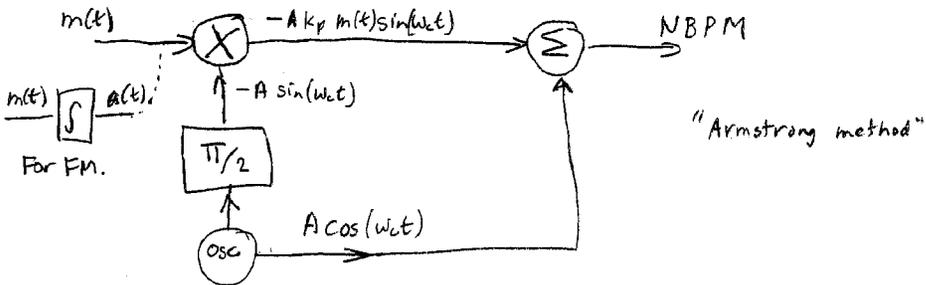


Can ignore all but the first 2 terms ---

$$\Phi_{FM}(t) \approx A [\cos(\omega_c t) - K_f a(t) \sin(\omega_c t)]$$

it looks like AM --

actually quadrature AM with carrier,



Wide-band

7C-2

doesn't satisfy $|K_f a(t)| \ll 1$.

$(|K_f m(t)| \ll \omega)$
↑ not satisfied.

Can't ignore higher terms.

Book approach:

Consider it to be a sampled system.

Sample at $T = \frac{1}{2B}$ (Nyquist rate).

You get an approximation based on constant-amplitude cells (sample + hold)

The FM signal is a sinusoid, for this time --

$$A \cos(\omega_c + K_f m(t_k))$$

↑ value in that cell.

Take Fourier transform, add them ----

Define frequency deviation: $\Delta f = \frac{K_f m_p}{2\pi}$

$$\Delta \omega = K_f m_p$$

$m_p =$ peak modulation amplitude

$$B_{FM} = \frac{1}{2\pi} (2K_f m_p + 8\pi B) \leftarrow \text{derivation not shown. "Trust me!"}$$

$$= 2 (\Delta f + 2B)$$

Deviation $m(t)$ bandwidth.

Bandwidth is determined by deviation (Δf) and modulating frequency ($2B$)

→ This approximation assumes $2B$ additional beyond deviation.

x2 because upper + lower...

"Carson's rule" ... above is pessimistic.

7C-3

$$B_{FM} \approx 2(\Delta f + B)$$

↑ one sideband beyond deviation.

$$B_{FM} = 2 \left(\frac{k_f m_p}{2\pi} + B \right)$$

(Hertz)

Define: "deviation ratio" $\beta = \frac{\Delta f}{B} = \frac{\text{deviation}}{\text{baseband-bandwidth}}$

"modulation index"

For wideband -- $\beta \gg 1$
($\Delta f \gg B$)

substitute:

$$B_{FM} = 2(\beta B + B) \\ = 2B(\beta + 1)$$

Phase modulation - similar, but --

$$w_i = w_c + k_p \dot{m}(t)$$

$$\Delta w = k_p \dot{m}'_p$$

↑ peak value of $\dot{m}(t)$

$$B_{PM} = 2(\Delta f + B) \quad (\text{Carson's rule})$$

$$B_{PM} = 2 \left(\frac{k_p m'_p}{2\pi} + B \right)$$

$\dot{m}(t)$ strongly depends on B
(differentiated)

Next time: more math.

Accurate analysis

7C-4

or.... We can really analyze it with tone modulation.
Let's try to do a detailed analysis for $m(t) = \alpha \cos(\omega_m t)$
which is the same as $A(t) = \frac{\alpha}{\omega_m} \sin(\omega_m t)$
(by the formula on 7B-2)

Plugging into $\hat{\phi}_{FM} = A e^{j[\omega_c t + k_f a(t)]}$

$$\hat{\phi}_{FM} = A e^{j[\omega_c t + k_f \frac{\alpha}{\omega_m} \sin(\omega_m t)]}$$

↑ book error - j applies here too.

Frequency deviation is --

$$\Delta w = k_f m_p = k_f \alpha$$

Bandwidth of $m(t)$ is --

$$B = f_m = 2\pi \omega_m$$

Modulation index is --

$$\beta = \frac{\Delta f}{f_m} = \frac{\Delta w}{\omega_m} = \frac{k_f \alpha}{\omega_m}$$

Plugging this in

$$\hat{\phi}_{FM}(t) = A e^{j[\omega_c t + \beta \sin(\omega_m t)]}$$

$$= A e^{j\omega_c t} e^{j\beta \sin(\omega_m t)}$$

periodic signal
period = $\frac{2\pi}{\omega_m}$

Expand in Fourier series--

Recall --- $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_m t}$ 7C-5

$$C_n = \frac{1}{T_0} \int_{T_0} g(t) e^{-jn\omega_m t} dt$$

$$g(t) = e^{j\beta \sin(\omega_m t)} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_m t}$$

look left this out.

$$C_n = \frac{\omega_m}{2\pi} \int_{-\frac{\pi}{\omega_m}}^{\frac{\pi}{\omega_m}} e^{j\beta \sin(\omega_m t)} e^{-jn\omega_m t} dt$$

Let $\omega_m t = x$

$$t = \frac{x}{\omega_m}$$

$$C_n = \frac{\omega_m}{2\pi} \int_{-\frac{\pi}{\omega_m}}^{\frac{\pi}{\omega_m}} e^{j\beta \sin(\omega_m \frac{x}{\omega_m})} e^{-jn\omega_m \frac{x}{\omega_m}} d\frac{x}{\omega_m}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\beta \sin(x)} e^{-jnx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin(x) - nx)} dx$$

This is a

"Bessel function of the first kind and nth order"

$$C_n = J_n(\beta)$$

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin(x) - nx)} dx$$

see plot - p. 222.
(strange axes).

(insert p. 325, 326 from couch)

Octave function;
besselj(n, x)

Example:

$$f_c = 100 \text{ MHz.}$$

7C-6

$$\Delta f = 75 \text{ kHz.}$$

$$f_m = 1, 10, 100$$

Find the spectrum, including all components $> .01$

This means: $\beta = 75, 7.5, .75$

$$C_n = J_n(\beta) \text{ -- octave-- besselj}(n, \beta)$$

100 kHz	n:	0	1	2	3			
	$J_n(\beta)$:	.864	.349	.067	.008			
10 kHz	n:	0	6	9	10	11		
	$J_n(\beta)$:	.266	.13541	.0889	.0389	.0052		
1 kHz	n:	0	71	72	80	81	82	83
	$J_n(\beta)$:	.0346	.158	.159	.0215	.014	.009	.008

Bandwidth containing all components $> .01$ --

f_m	SB#	one side	total	Carson's rule	formula 7C-2
100K	2	200K	400K	350K	550K
10K	10	100K	200K	170K	190K
1K	81	81K	162K	152K	154K

