efficiency, which relates to concentrating the energy at low frequencies, ease of computation, and minimum mean square error. The ideal transform for achieving these is the Karhunen–Loève transform, but this cannot be represented algorithmically. However, the discrete cosine transform (DCT) has virtually the same properties and does possess an algorithm. It consists essentially of the real part of the DFT. This definition is reasonable since the Fourier series of a real and even function contains only the cosine terms, and in, for example, the case of sampled voltage values the data used is real and can be made symmetrical by doubling the data by adding its mirror image. Thus the DFT is given by (Equation 3.41)

$$X(k) = \sum_{n=0}^{N-1} x_n e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots, N-1$$

Defining the DCT $X_c(k)$ as the real part of this gives

$$X_c(k) = \text{Re}\,[X(k)] = \sum_{n=0}^{N-1} x_n \cos\left(\frac{k2\pi n}{N}\right), \quad k = 0, 1, \ldots, N-1 \tag{3.66}$$

This is one of several forms of the DCT. A more common form is (Beauchamp, 1987; Yip and Ramamohan, 1987; Ahmed and Rao, 1975)

$$X_c(k) = \frac{1}{N}\sum_{n=0}^{N-1} x_n \cos\left(\frac{k2\pi n + k\pi}{2N}\right) = \frac{1}{N}\sum_{n=0}^{N-1} x_n \cos\left[\frac{k\pi(2n+1)}{2N}\right] \qquad k = 0, 1, \ldots, N-1 \tag{3.67}$$

and other forms also exist (Yip and Ramamohan, 1987; Pennebaker and Mitchell, 1993; Pitas, 1993; Bailey and Birch, 1989).

Implementations of the DCT exist based on the FFT as might be expected (Narasinka and Petersen, 1978), and a fast DCT which is six times as fast as these has been developed (Chen *et al.*, 1977). Another version is the C-matrix transform which can be more simply constructed in hardware (Srinivassan and Rao, 1983).

3.8.2 Walsh transform

The transforms discussed so far have been based on cosine and sine functions. Transforms based on pulse-like waveforms which take only values of ±1 are much simpler and faster to compute. They are also more appropriate for the representation of waveforms which contain discontinuities, for example in images. Conversely, they are less appropriate for describing continuous waveforms and may not be phase invariant in which case the derived spectrum may be distorted. However, such waveforms are used in image processing (astronomy and spectroscopy), signal coding, and filtering.

Just as the DFT is based on a set of harmonically related cosine and sine waveforms, so is the discrete Walsh transform (DWT) based on a set of harmonically related rectangular waveforms, known as Walsh functions. However, frequency is not

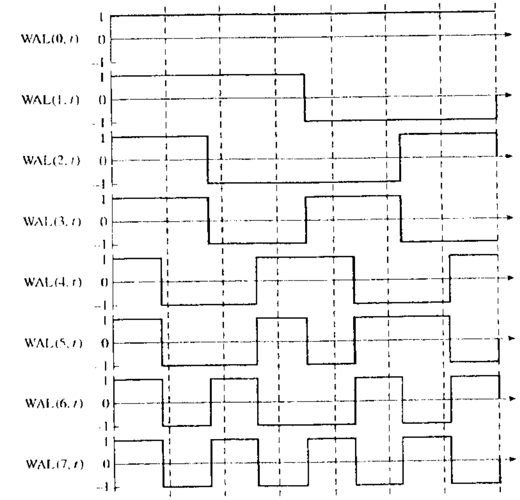

Figure 3.6 Sequency-ordered Walsh functions to $n = 7$ showing sampling times for 8 × 8 Walsh transform matrix.

defined for rectangular waveforms and so the analogous term sequency is used. Sequency is half the average number of zero crossings per unit time. Figure 3.6 shows the set of Walsh functions up to the order of $N = 8$ drawn in order of increasing sequency. They are said to be sequency, or Walsh, ordered. The Walsh function at time t and of sequency n is designated WAL(n, t). Inspection of Figure 3.6 shows that there are equal numbers of even and odd Walsh functions, just as there are corresponding cosinusoidal and sinusoidal Fourier series components. The even functions, WAL($2k$, t), are written CAL(k, t), and the odd functions, WAL($2k+1$, t), are written SAL(k, t), where $k = 1, 2, \ldots, N/2 - 1$.

Any waveform, $f(t)$, may be written as a Walsh function series, analogous to a Fourier series, as

$$f(t) = a_0\,\text{WAL}(0, t) + \sum_{i=1}^{N/2-1}\sum_{j=1}^{N/2-1}[a_i\,\text{SAL}(i, t) + b_j\,\text{CAL}(j, t)] \tag{3.68}$$

where the a_i and b_j are the series coefficients.

For any two Walsh functions,

$$\sum_{i=0}^{N-1} \mathrm{WAL}(m, t)\,\mathrm{WAL}(n, t) = \begin{cases} N & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases}$$

that is, Walsh functions are orthogonal.

The discrete Walsh transform pair is

$$X_k = \frac{1}{N} \sum_{i=0}^{N-1} x_i \,\mathrm{WAL}(k, i) \quad k = 0, 1, \ldots, N-1 \tag{3.69}$$

and

$$x_i = \sum_{i=0}^{N-1} X_k \,\mathrm{WAL}(k, i) \quad i = 0, 1, \ldots, N-1 \tag{3.70}$$

where we note that, apart from the factor of $1/N$, the inverse transform is the same as the transform, and that $\mathrm{WAL}(k, i) = \pm 1$. The transform pair may, therefore, be calculated by matrix multiplication by digital means as mentioned above. However, the lack of phase invariance means that the DWT is unsuitable for fast correlations or convolutions.

Equation 3.69 shows that the kth DWT component is obtained by multiplying each waveform sample x_i by the Walsh function of sequency k and summing for $k = 0, 1, \ldots, N-1$. This may be expressed for all k DWT components in matrix notation as

$$\mathbf{X}_K = \mathbf{x}_i \mathbf{W}_{ki} \tag{3.71}$$

where $\mathbf{x}_i = [x_0\ x_1\ x_2 \ldots x_{N-1}]$, the data sequence,

$$\mathbf{W}_{ki} = \begin{bmatrix} W_{01} & W_{02} & \cdots & W_{0,N-1} \\ W_{11} & & & \\ \vdots & & & \vdots \\ W_{N-1,1} & W_{N-1,2} & \cdots & W_{N-1,N-1} \end{bmatrix}$$

the Walsh transform matrix, and $\mathbf{X}_k = [X_0\ X_1 \ldots X_{N-1}]$, the $N-1$ components of the DWT. Note that $\mathbf{W}_{ki}$ is an $N \times N$ matrix where N is the number of data points, that is sampled waveform points. Thus, if there are N data points it is necessary to consider the first N sequency-ordered Walsh functions. Each one is sampled N times. The kth row of $\mathbf{W}_{ki}$ corresponds to the N sampled values of the kth sequency component.

Example 3.6 As an example, let us compute the DWT of the data sequence (1, 2, 0, 3). This consists of $N = 4$ data points and so $\mathbf{W}_{ki}$ is a 4×4 matrix obtainable from the first four rows of Figure 3.6 as

$$\mathbf{W}_{ki} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \tag{3.72}$$

Therefore, from Equation 3.71, $\mathbf{X}_k$ is given by

$$\mathbf{X}_k = \tfrac{1}{4}[1\quad 2\quad 0\quad 3] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \tfrac{1}{4}[6\quad 0\quad 2\quad -4]$$

so that $X_0 = 1.5$, $X_1 = 0$, $X_2 = 0.5$ and $X_3 = -1$. This is considerably easier to calculate than the corresponding DFT! Needless to say, fast DWTs (FDWTs) exist.

The corresponding spectrum can be calculated with power components given by

$$P(k) = [|\mathrm{CAL}(k, t)|^2 + |\mathrm{SAL}(k, t)|^2]^{1/2}$$

where

$$\begin{aligned} P(0) &= X_c^2(0) \\ P(k) &= X_c^2(k, t) + X_s^2(k, t) \\ P\left(\frac{N}{2}\right) &= X_s^2\left(\frac{N}{2}, t\right) \end{aligned} \tag{3.73}$$

where $k = 1, 2, \ldots, N/2 - 1$, and with phase components

$$\begin{aligned} \phi(0) &= 0, \pi \\ \phi(k) &= \tan^{-1}\left[\frac{X_s(k)}{X_c(k)}\right], \quad k = 1, 2, \ldots, N/2 - 1 \end{aligned} \tag{3.74}$$

and

$$\phi\left(\frac{N}{2}\right) = 2k\pi \pm \pi/2, \quad k = 0, 1, 2, \ldots$$

For the above DWT we have, therefore

$$P(0) = 1.5^2 = 2.25;\ \phi(0) = 0, \pi$$

$$P(1) = 0^2 + 0.5^2 = 0.25;\ \phi(1) = \tan^{-1}\left(\frac{0}{0.5}\right) = 0$$

$$P(2) = (-1)^2 = 1;\ \phi(2) = \frac{\pi}{2} + 2k\pi, \quad k = 0, 1, 2$$

3.8.3 Hadamard transform

The Hadamard transform, or Walsh–Hadamard transform, is basically the same as the Walsh transform, but with the Walsh functions and therefore the rows of the transform