More Fourier series

~~HW 2.8 4 2.9 1~~

Last time — introduced it
did an example.

This time — details
Exponential F.S.
Fourier integral ???

Reminder — "Compact" trigonometric Fourier Series

see 1D-11

This is the usual form —
it gives frequency and phase

Putting our example into compact form;

$C_n = \sqrt{a_n^2 + b_n^2}$ — since all $a_n = 0$

$C_n = b_n$

$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$ since all $a_n = 0$

$\tan^{-1}(\infty) = -\frac{\pi}{2}$ radians

$= -90$ degrees

$$\frac{1}{2} + \frac{2}{\pi}\cos\left(\pi t - \frac{\pi}{2}\right) + \frac{2}{3\pi}\cos\left(3\pi t - \frac{\pi}{2}\right)$$

$$+ \frac{2}{5\pi}\left(5\pi t - \frac{\pi}{2}\right) + \cdots$$

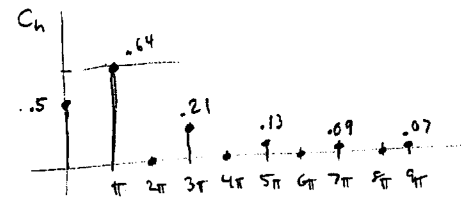
$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\cos\left(n\pi t - \frac{\pi}{2}\right)$$

odd only

This π refers to the phase angle.

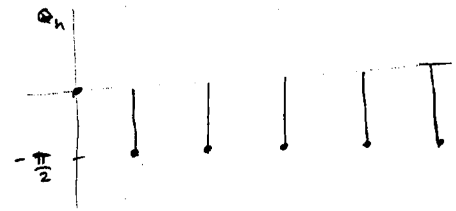
This π is a coincidence $\omega_0 = \pi$ here.

Amplitude & phase plot:



π is the fundamental

3π, 5π, etc are harmonics.



Properties:

Periodicity —

We found the Fourier series over one period T_0.

In this case — $T_0 = 2$

So the fundamental $= f = \frac{1}{T_0} = \frac{1}{2}$ Hz

$\omega = \frac{2\pi}{T_0} = \frac{2\pi}{2} = \pi \frac{\text{rad}}{\text{sec}}$

We only looked at the time interval T_0.

The Fourier series results in a periodic signal, period T_0 (or nT_0)

What if we let $T_0 = 4$?
What would the result be?

Fourier spectrum

We can plot amplitude (C_n) and phase (θ_n) vs. Frequency

This is a "frequency domain description" of the signal

Dual identity —

A signal has a dual identity — it can be expressed in either frequency or time domain.

Convergence at jump discontinuities

It converges to the midpoint

$\leftarrow$ converges to $\frac{13-5}{2} = 8$

Existence

① For the series to exist, the coefficients must be finite:

$$\int_{T_0} |g(t)|\, dt < \infty$$

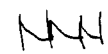
("weak Dirichlet Condition")

$\rightarrow$ but that doesn't guarantee convergence

Convergence means that higher harmonics approach zero.

② For the series to converge, $g(t)$ must have a finite number of minima and maxima and a finite number of discontinuities — in one period ("strong Dirichlet condition")

All real signals meet both criteria.

More signals — see text p 46 
50 another square wave
52 impulses.

Exponential Fourier Series

Basis functions are $e^{jn\omega_0 t}$ $(n = 0, \pm 1, \pm 2, \ldots)$

(same as $\cos n\omega_0 t + j \sin n\omega_0 t$)

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \left(\sum_{n=-\infty}^{\infty} c_n x(t) \right)$$

$$D_n = \frac{1}{T_0} \int_{T_0} g(t) e^{-jn\omega_0 t} dt \qquad \left(\frac{1}{T_0} \int_{T_0} g(t) x^*(t)\, dt \right)$$

where $x(t) = e^{jn\omega_0 t}$

so $x^*(t) = e^{-jn\omega_0 t}$

This one, specific.

Recall general definition of Fourier series

- equivalent to The Trig. Fourier series.
- D_n are complex. – but we usually use polar form.
- More convenient to calculate, so more common.

Negative frequency?

Artifact of the representation.

It says the same thing again.

$$\cos(-\omega_0 t + \theta) = \cos(\omega_0 t - \theta)$$

A frequency is expressed as a pair of exponentials – $e^{\pm jn\omega_0 t}$

Amplitude of the negative frequency is the same

Phase is minus.

or ----- Value is conjugate of the positive value.

Numerical Computation

Discrete Fourier Transform – not covered here,

but MATLAB / octave have it

Most simulators have it.

(Gnucap "fourier" command)

Spice Fourier command is lame.

Chapter 3 —

Analysis + Transmission of Signals. -

Fourier Integral / Transform

For Aperiodic signals.

Just let the length of Time approach ∞ coeff. basis fcn

(All signals are periodic if you wait long enough)

Recall - - - $g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$ function but approximated basis function

These are the useful \$\$ → $D_n = \frac{1}{T_0}\int_{T_0} g_{T_0}(t)\, e^{-jn\omega_0 t}\, dt$

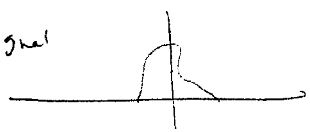
(Fourier Series)

Let $T_0 \to \infty$

$$D_n = \frac{1}{T_0}\int_{-\infty}^{\infty} g(t)\, e^{-jn\omega_0 t}\, dt$$

$\uparrow$ $\frac{1}{\infty}$?

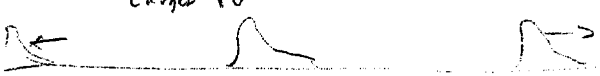
signal



make periodic

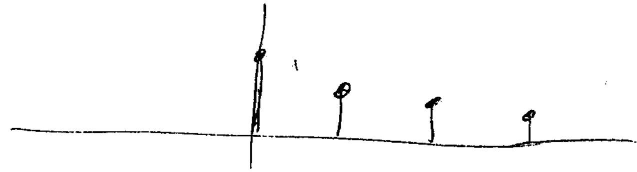
Larger T_0

Larger To

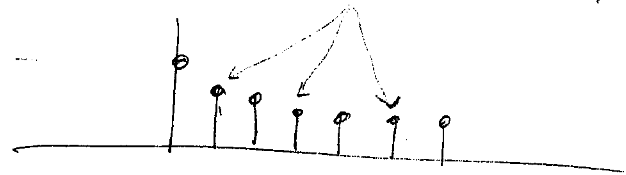


What are we doing here?

Fourier series gives us discrete frequencies.



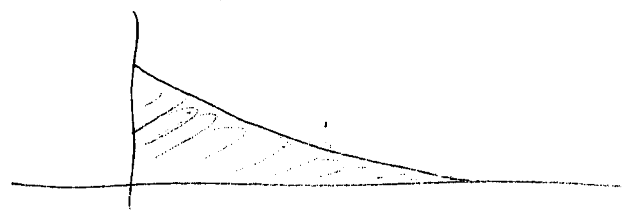
Doubling the period gives us more — lower and closer spaced.



The lowest frequency and spacing $= \frac{1}{T_0}$

By letting $T_0 \to \infty$, $\frac{1}{T_0} \to 0$

approaches a continuous spectrum



continuous D_n (C_n -- or whatever) now continuous

Define: $G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$ — This is "Fourier transform" so what?

so $D_n = \frac{1}{T_0} G(n\omega_0) = \frac{G(n\omega_0)}{T_0}$ ← n = discrete frequencies

Substitute into Fourier Series

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{G(n\omega_0)}{T_0} e^{jn\omega_0 t}$$

Replace ω_0 by $\Delta\omega$ — spacing between freqs. $\Delta\omega = \frac{2\pi}{T_0}$

lowest freq

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{G(n\Delta\omega)\Delta\omega}{2\pi} e^{jn\Delta\omega t}$$

Now let $T_0 \to \infty$ $(\Delta\omega \to 0)$

$$g(t) = \lim_{T_0\to\infty} g_{T_0}(t) = \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} \frac{G(n\Delta\omega)\Delta\omega}{2\pi} e^{jn\Delta\omega t}$$

rearrange →

$$= \underbrace{\lim_{\Delta\omega\to 0} \frac{1}{2\pi} \underbrace{\sum_{n=-\infty}^{\infty} G(n\Delta\omega) e^{jn\Delta\omega t} \Delta\omega}_{\text{This is area under } G(\omega)e^{j\omega t}}}_{\text{looks like an integral}}$$

$$\underbrace{g(t) = \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega}_{\text{This is "Fourier integral"}}}_{\text{This is "inverse Fourier transform"}}$$

This gives us:

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$ ← "direct"

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{+j\omega t} d\omega$$ ← "inverse"

↑ Fourier transform pair