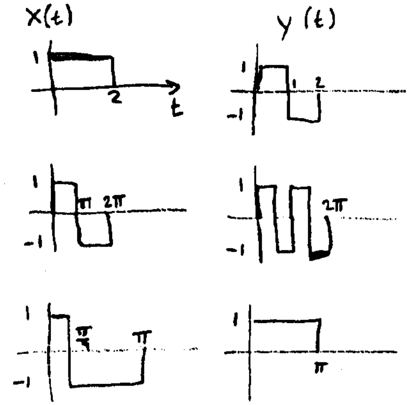
Homework --
Ch. 2 (p. 115)
basics – 2, 4, 5, 6, 7, 8, 9, 10

Find $E_x, E_y$. Show that $E_{(x+y)} \overset{?}{=} E_x + E_y$, $E_{(x-y)} \overset{?}{=} E_x + E_y$ (or not ...)

(a)

(b)

(c)




Find the correlation coefficient
between: $x(t) = \sin 2\pi t$ $E_x = .5$

and $g_1(t) = \sin 4\pi t$ $E_{g_1} = .25$

$g_2(t) = -\sin 2\pi t$ $E_{g_2} = .5$

$g_3(t) = .707$ $E_{g_3} = .5$

$g_4(t) = \begin{cases} .707 & 0 .. .5 \\ -.707 & .5 .. 1 \end{cases}$ $E_{g_4} = .5$

$g(t) = t$ over $(-\pi, \pi)$, is periodic with period $2\pi$
Sketch $g(t)$
Find the Fourier coefficients, for 10 % error.

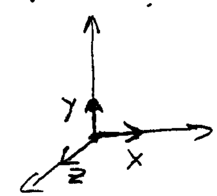


## Signal representation by orthogonal signal set

Given any signal,
we can represent it as the sum of
orthogonal basis components.

First, look at vectors (2, 7, 1)

Consider a 3-dimensional space,
with 3-mutually orthogonal basis vectors
X, Y, Z




– we have a coordinate system –
the basis vectors are of unit length
in the direction of an axis.

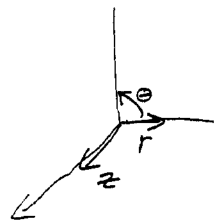
Consider a vector from the origin
to a point (3, 5, 2).

We can represent it as $g = 3x + 5y + 2z$

where X, Y, Z are orthogonal basis vectors.

The $x, y, z$ form is common,
Actually we could use any set of mutually orthogonal vectors.

$r, \theta, z$




where $r = \sqrt{x^2 + y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$

$$g = 5.83\,r + 59.04\,\theta + 2z$$

Change of basis is an important concept in signal processing, (Color TV).

Video: R, G, B
Transmitted: Y, I, Q

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If the set of basis vectors is not complete, there will be some error -

$$g \approx c_1 x + c_2 y$$

$$e = g - (c_1 x + c_2 y)$$

↑ error

(missing $z$ component)



<u>Completeness</u> means that it is impossible to find another vector ($x_4$) that is orthogonal to all that we have. ($x, y, z$)

A set of vectors ($X_i$) is mutually orthogonal if:

$$X_m \cdot X_n = \begin{cases} 0 & m \neq n \\ |X_n|^2 & m = n \end{cases}$$

If the basis set is complete —
$g$ can be expressed as the sum;

$$g = \sum c_i x_i$$

and coefficients are:

$$c_i = \frac{g \cdot x_i}{x_i \cdot x_i} \qquad i = 1, 2, 3, \ldots$$

## Orthogonal Signal space

Instead of using a sum (vectors)
use an integral --
inner product instead of dot product, etc.

Backup --- define Orthogonality in complex signals: (2.5,3)

Recall..

Energy: $E_x = \int_{t_1}^{t_2} |x(t)|^2 \, dt$

Error: $e(t) = g(t) - c\,x(t)$

$$E_e = \int_{t_1}^{t_2} |g(t) - c\,x(t)|^2 \, dt$$

Math-identity:

$$|u+v|^2 = (u+v)(u^* + v^*)$$

conjugate

$$= |u|^2 + |v|^2 + u^* v + u v^*$$

Substituting & solving --

$$c = \frac{1}{E_x} \int_{t_1}^{t_2} g(t)\, x^*(t)\, dt$$

conjugate

Orthogonality is defined as:

$$\int_{t_1}^{t_2} x_1(t)\, x_2^*(t)\, dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2} x_1^*(t)\, x_2(t)\, dt = 0$$

contrast to $\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt$ for real signals.



back to orthogonal signal space...

We want to represent our signal $g(t)$
as a sum of orthogonal signals $x_1(t), x_2(t), x_3(t), \ldots$
over the time interval $[t_1, t_2]$..

The signals $x_n$ are orthogonal if:

$$\int_{t_1}^{t_2} x_m(t)\, x_n^*(t)\, dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

If all energies $E_n = 1$, the set is normalized and is called an orthonormal set.

We can make an orthonormal set by dividing all $x_n(t)$ by $\sqrt{E_n}$.

So ----

$$g(t) \approx c_1 x_1(t) + c_2 x_2(t) + \ldots\ldots$$

$$= \sum_{n=1}^{N} c_n x_n(t) \qquad t_1 \le t \le t_2$$

$$c_n = \frac{\int_{t_1}^{t_2} g(t)\, x_n^*(t)\, dt}{\int_{t_1}^{t_2} x_n^2(t)\, dt}$$

$$= \frac{1}{E_n} \int_{t_1}^{t_2} g(t)\, x_n^*(t)\, dt \qquad n = 1, 2, \ldots$$

Remember ...
$g(t)$ is the signal
$x_n$ are a set of orthonormal basis signals

If the orthogonal set is complete
error energy $= 0$
and the representation is exact:

$$g(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t)$$

now =.

(but we rarely get this)

More typically .... $c_N$ gets smaller as $N \to \infty$

This is a generalized Fourier series

Specific cases:

Let $x_n$ = sines and cosines

$= \{1, \cos\omega_0 t, \sin\omega_0 t,$
$\cos 2\omega_0 t, \sin 2\omega_0 t,$
$\cos 3\omega_0 t, \sin 3\omega_0 t,$
$\ldots\}$

→ Trigonometric Fourier Series

Let $x_n$ = exponentials.

$= e^{jn\omega_0 t} \quad n = 0, \pm 1, \pm 2, \ldots$

→ Exponential Fourier series.



Others that we use:

Chebyshev polynomials — Filters
Bessel functions — FM
Walsh functions — data compression

Parseval's theorem —

The energy of a signal equals the sum of the energies in all its components.

$$E_g = c_1^2 E_1 + c_2^2 E_2 + c_3^2 E_3 + \ldots$$
$$= \sum_n c_n^2 E_n$$

Energy of a component is:

$c_n^2 E_n$

the coefficient

Energy in the basis function

Kirchoff's laws are special cases of Parseval's theorem.

# Trigonometric Fourier Series –

use sines + cosines - - -

$n\omega_0$ is the "n th harmonic" of $\omega_0$

$n$ is an integer

$\omega_0$ is the fundamental

other terms are harmonics

$\{ 1,$ ← 0 th harmonic

actually - - - $\cos(0) = 1$

= D C component.

$\cos \omega_0 t, \sin \omega_0 t,$

$\cos 2\omega_0 t, \sin 2\omega_0 t,$

$\dots \}$

$$g(t) = a_0 + a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t) + a_2 \cos(2\omega_0 t) + b_2 \sin(2\omega_0 t) + \dots$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$t_1 \le t \le t_1 + \underbrace{T_0}_{\text{some time interval.}}$$



To determine the coefficients - - -

$$a_n = \frac{\int_{t_1}^{t_1+T_0} g(t) \cos(n\omega_0 t)\, dt}{\int_{t_1}^{t_1+T_0} \cos^2(n\omega_0 t)\, dt}$$

(from p. 6)

The denominator is $\frac{T_0}{2}$ for $n = 1 \dots$

and $T_0$ for $n = 0$

So it can be simplified - -

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} g(t)\, dt$$

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} g(t) \cos(n\omega_0 t)\, dt \qquad n = 1, 2, 3, \dots$$

likewise - -

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} g(t) \sin(n\omega_0 t)\, dt \qquad n = 1, 2, 3, \dots$$

"Compact" trigonometric Fourier series –

A shorter way to write the same thing;

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = C_n \cos(n\omega_0 t + \theta_n)$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

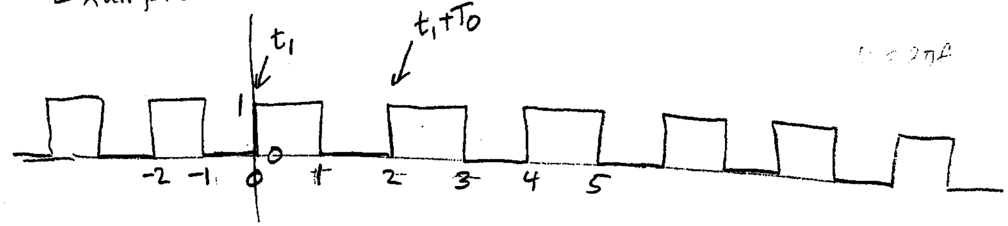
$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

$$C_0 = a_0$$

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$



Example:




$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} g(t)\,dt$$

$T_0 = 2$ $\quad f_0 = \frac{1}{2}$

$\omega = \frac{2\pi}{2} = \pi$

$$= \frac{1}{2}\Big[\underbrace{\int_0^1 1\,dt}_{1} + \underbrace{\int_1^2 0\,dt}_{0}\Big]$$

$$= \frac{1}{2}$$

$$a_1 = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} g(t)\cos(\omega_0 t)\,dt$$

$$= \underbrace{\frac{2}{2}}_{1}\Big[\int_0^1 1\cos(\pi t)\,dt + \underbrace{\int_1^2 0\cos(\pi t)\,dt}_{0}\Big]$$

$$= \frac{1}{\pi}\sin(\pi t)\Big|_1 - \frac{1}{\pi}\sin(\pi t)\Big|_0$$

$$= \sin\pi - \sin 0$$

$$= 0$$

$$b_1 = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} g(t)\sin(w_0 t)\,dt$$

$$= \underbrace{\frac{2}{2}}_{1}\left[\int_0^1 1\,\sin(\pi t)\,dt + \underbrace{\int_1^2 0\,\sin(\pi t)\,dt}_{0}\right]$$

$$= \frac{1}{\pi}\cos(\pi t)\Big|_1 - \left(-\frac{1}{\pi}\cos(\pi t)\right)\Big|_0$$

$$= \frac{1}{\pi}\cos\pi + \cos 0$$

$$= -\left(-\frac{1}{\pi}\right) + \frac{1}{\pi}$$

$$= \frac{2}{\pi} = .64$$

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more ...

$a_2 = 0$ $\quad b_2 = 0$

$a_3 = 0$ $\quad b_3 = \frac{2}{3}\pi$

$a_n = 0$ $\quad b_n = 0$ for $n$ even

$b_5 = \frac{2}{5}\pi$

all sine terms.

even harmonics are 0



So, this signal is:

$$\overset{0}{\frac{1}{2}} + \overset{1}{\frac{2}{\pi}\sin\pi t} + \overset{3}{\frac{2}{3\pi}\sin 3\pi t} + \overset{5}{\frac{2}{5\pi}\sin 5\pi t}$$

$$+ \frac{2}{7\pi}\sin 7\pi t + \cdots\cdots$$