Today: <u>Correlation</u> (2.5, 2.6)

What is it?

Suppose we have two signals,

or one real signal and a "reference"

We want to know:

Are they the same?

if not, how different are they?

So — are they the same, except for some noise and distortion?

The approach --

A signal is a <u>vector</u>, so look at some properties of vectors.

Extend them to cover signals, which are multi-dimensional

<u>Signals + Vectors.</u> (2.5)

Signals are vectors.

Simple: vectors have magnitude and direction.

More complex: Space is n-dimensional.

<u>Component of a vector</u> -

We want to represent vectors by components:

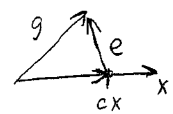
g is a vector —

x is another vector

signals context: g is one signal, x is another

we can make a projection of g onto x

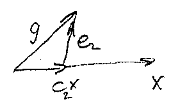
call it "$c\,x$" c is a constan



The vector from the end of the projection to the end of g is the "error vector".

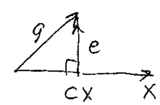
$$g = cx + e$$

We could have made a different projection:



$$g = c_2 x + e_2$$

Ideally, we choose c to minimize e.



We would like to think that:

$$g \approx cx$$

In this context --

$\underline{c}$ is a constant multiplier (scale factor)

We choose the value to minimize the error vector (e).

That will happen when e is orthogonal to x.

Define: dot product:

$$g \cdot x = |g||x| \cos \theta$$

length of ---

θ is the angle between vectors

when -- $\theta = 0$, $\cos\theta = 1$

g and x are the same direction

$\theta = 90^\circ$, $\cos\theta = 0$

g and x are orthogonal — no relation to each other.

We can express $|x|$ using this definition:

$$|x|^2 = x \cdot x$$

The length of the component g along x

is $c|x|$

it is also $|g| \cos\theta$

so-- $c|x| = |g|\cos\theta$

Multiply both sides by $|x|$

$$c|x|^2 = \underbrace{|g||x| \cos\theta}_{\text{this is } g \cdot x}$$

so ..

$$c|x|^2 = g \cdot x$$

Solve for c:

$$c = \frac{g \cdot x}{|x|^2} = \frac{g \cdot x}{x \cdot x}$$

When g and x are perpendicular (orthogonal) then g has a zero component along x.

so $c = 0$

g and x are orthogonal if the dot product is (

The dot product is an indication of how alike x and g are.

Extend this to signals - (2.5.2)

We want to approximate a signal $g(t)$ in terms of another signal $x(t)$ over an interval $[t_1, t_2]$

Hopefully: $g(t) \approx c\, x(t) \qquad t_1 \leq t \leq t_2$

The error is

$$e(t) = \begin{cases} g(t) - c\, x(t) & t_1 \leq t \leq t_2 \\ 0 & \text{otherwise} \end{cases}$$

It isn't defined here, so there can't be any error.

We need some way to judge the "best approximation"

Use the energy in the error..

$$E_e = \int_{t_1}^{t_2} e^2(t)\, dt$$

$$= \int_{t_1}^{t_2} [g(t) - c\, x(t)]^2\, dt$$

To find the minimum E (energy), set the derivative to 0. (with respect to $\underline{c}$)

$$\frac{dE_e}{dc} = 0$$

Substitute:

$$\frac{d}{dc}\left[\int_{t_1}^{t_2} [g(t) - c\, x(t)]^2\, dt\right] = 0$$

Expand the squared term:

$$\underbrace{\frac{d}{dc}\left[\int_{t_1}^{t_2} g^2(t)\, dt\right]}_{0} - \underbrace{\frac{d}{dc}\left[2c\int_{t_1}^{t_2} g(t)\, x(t)\, dt\right]}_{2\int_{t_1}^{t_2} g(t)\, x(t)\, dt} + \underbrace{\frac{d}{dc}\left[c^2\int_{t_1}^{t_2} x^2(t)\, dt\right]}_{2c\int_{t_1}^{t_2} x^2(t)\, dt} =$$

$$0 - 2\int_{t_1}^{t_2} g(t)\, x(t)\, dt + 2c\int_{t_1}^{t_2} x^2(t)\, dt = 0$$

Solve for c

$$c = \frac{\int_{t_1}^{t_2} g(t)\, x(t)\, dt}{\int_{t_1}^{t_2} x^2(t)\, dt}$$

This is E_x (energy in x)

$$= \frac{\int_{t_1}^{t_2} g(t)\, x(t)\, dt}{E_x}$$

this is called "inner product"

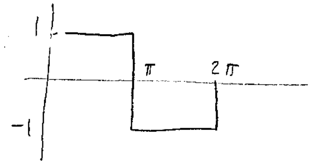
Now we can say that $c\, x(t)$ is the best approximation along $x(t)$ of $g(t)$.

If the numerator is 0, x and g are orthogonal. (uncorrelated)

Example:

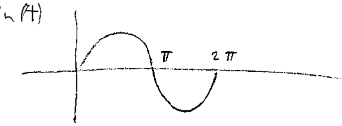
Given a square signal $g(t)$

$$g(t) = \begin{cases} 1 & 0 \le t < \pi \\ -1 & \pi \le t < 2\pi \end{cases}$$



Approximate it in terms of $\sin(t)$

$$x(t) = \sin(t)$$



so the energy of the error signal is minimum

Use the formula: $$c = \frac{\int_{t_1}^{t_2} g(t)\, x(t)\, dt}{\int_{t_1}^{t_2} x^2(t)\, dt} \qquad \left(= \frac{(g, x)}{E_x} \right)$$

Denominator: $$E_x = \int_0^{2\pi} \sin^2(t)\, dt = \pi$$

Numerator: $\longrightarrow$ $$\int_0^{\pi} \sin(t)\, dt + \int_{\pi}^{2\pi} -\sin(t)\, dt$$

$$= -\cos(t)\Big|_{\pi} - (-\cos(t))\Big|_{0} + \cos(t)\Big|_{2\pi} - (\cos(t))\Big|_{\pi}$$

$$= -\cos(\pi) + \cos(0) + \cos(2\pi) - \cos(\pi)$$

$$= 1 + 1 + 1 + 1$$

$$= 4$$

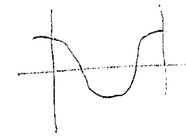
$$c = \frac{4}{\pi}$$

$$g(t) = \frac{4}{\pi} \sin(t)$$

Example 2

Approximate it in terms of $\cos(t)$

$$x(t) = \cos(t)$$



Denominator: $$E_x = \int_0^{2\pi} \cos^2(t)\, dt = \pi$$

Numerator: $$\int_0^{\pi} \cos(t)\, dt + \int_{\pi}^{2\pi} -\cos(t)\, dt$$

$$= \sin(t)\Big|_{\pi} - \sin(t)\Big|_{0} + (-\sin(t))\Big|_{2\pi} - (-\sin(t))\Big|_{\pi}$$

$$= \sin(\pi) - \sin(0) - \sin(2\pi) + \sin(\pi)$$

$$= 0 - 0 - 0 + 0$$

$$= 0$$

$c = 0$? ? ? ?

$\rightarrow$ signals are orthogonal.

Now -- Correlation (2.6)

or -- how similar are two signals?

Need a measure that is independent of the size (amplitude) of a signal.

Vectors ----

Two vectors $\underline{g}$ and $\underline{x}$ are similar if $\underline{g}$ has a large component along $\underline{x}$

With vectors -- the angle is a good indicator.

$\cos\theta$ is better -- it's in the range of -1 to 1

1 = vectors are ~~the same~~ aligned

-1 = - - - - - - opposite direction

0 = orthogonal

$$C_n = \cos\theta = \frac{g \cdot x}{|g||x|}$$ ← "correlation coefficient"

For signals--

need to consider entire time interval,

need a normalized version of "c"

The formula we used is relative to the magnitude of x

we need ~~to~~ normalize to both--

$$C_n = \frac{\int_{-\infty}^{\infty} g(t)\,x(t)\,dt}{\sqrt{E_g E_x}}$$

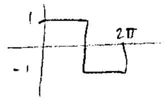
← same numerator

← normalize energies.

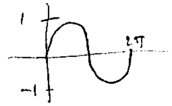
→ multiplying either $g(t)$ or $x(t)$ by a constant has no effect on C_n

Example --

(1) $g(t) =$



$x(t) =$



$$E_x = \int_0^{2\pi} \sin^2(t)\,dt = \pi \qquad E_g = \int_0^{\pi} (1)^2\,dt + \int_{\pi}^{2\pi} (-1)^2\,dt = 2\pi$$

Numerator = 4 (see p.7)

$$C_n = \frac{4}{\sqrt{(\pi)(2\pi)}} = \frac{4}{\pi\sqrt{2}} = \frac{4}{4.44} = .9$$

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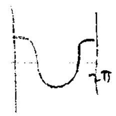
(2) $x(t) =$



Numerator $= 0$ $\qquad C_n = 0$