1

Higher order filters and design techniques —

We need to design to a spec. —

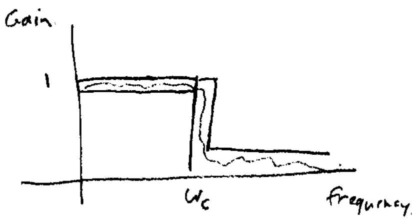
Example: $\quad w_c = 1000$ radians

Loss $= +0, -3$ dB in pass band.

Loss > 40dB for $w > 10000$

How do we do this in general?

Idea: Think of only a low pass for now ...

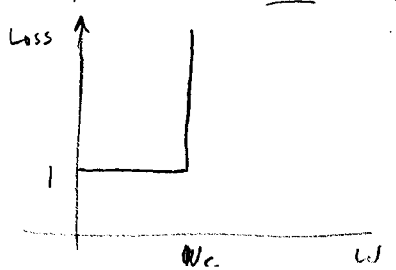


We really want this response.

Low pass transfer function: $$\frac{1}{a s^N + b s^{N-1} + \cdots\cdots +1}$$

A rational function with a polynomial in the denominator —

Think of loss instead of gain. $\quad$ Loss $= \frac{1}{\text{Gain}}$



Now we have a simple polynomial ---

$$a s^N + b s^{N-1} \cdots\cdots +1$$

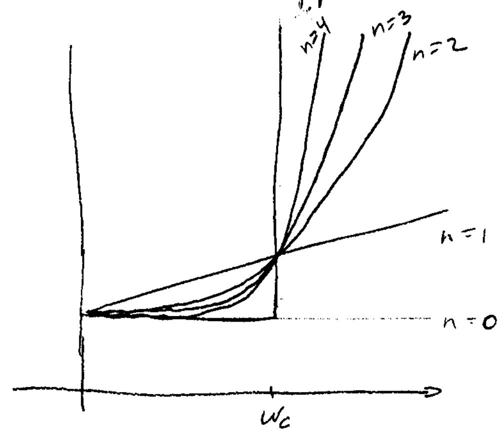
$\rightarrow$ a polynomial.

2

What is the "best" polynomial to fit the desired response?

How do we define "best"?

"Butterworth" approach

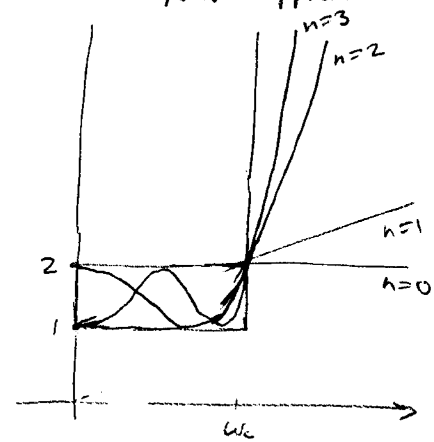


"Maximally flat"

No ripples in passband

More gradual transition

"Chebychev" approach



Passband has ripples

Steeper transition

Butterworth approximation

Optimal with respect to flatness of $H(j\omega)$
(No ripples!)

Flatness means $\frac{d|H|^2}{d\omega} = 0$ @ $\omega = 0$

1st derivative $\rightarrow$ $(2N+1)$ derivative
all $= 0$ @ $\omega = 0$

(slope, slope of slope, etc.)

where ω is radian frequency
N is the "order" of the polynomial.

The formula (without deriving it)

$$H(j\omega) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}}}$$

This gives an N'th order polynomial
where ω_c is the 3 dB frequency

Also often shown in "squared" form:

$$\underbrace{|H(j\omega)|^2}_{H(j\omega)\,H(-j\omega)} = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}}$$

To find the transfer function,
substitute $\omega = \frac{s}{j}$

which gives us:

$$\underbrace{|H(j\omega)|^2}_{= H(s)\,H(-s)} = \frac{1}{1 + \left(\frac{s}{j\omega_c}\right)^{2N}}$$

Normalizing to $\omega_c = 1$ ....

$$= \frac{1}{1 + \left(\frac{s_n}{j}\right)^{2N}} \qquad \leftarrow \left(\frac{1}{j}\right)^{2N} = (-1)^N$$

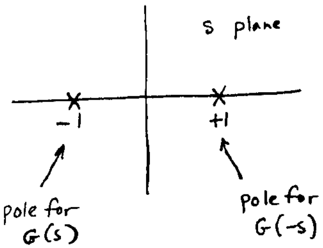
$$= \frac{1}{1 + (-1)^N (s_n)^{2N}}$$

$\rightarrow$ There are $2N$ poles
because N are associated with $H(s)$ ← left half plane
and N are associated with $H(-s)$ ← right half plane

Pole locations for $N=1$

$$G(s)\,G(-s) = \frac{1}{1+(-1)^1 (s)^2} = \frac{1}{1-s^2} = \frac{1}{(1+s)(1-s)}$$

$s=-1$ $\quad$ $s=1$



$$G(s) = \frac{1}{s+1}$$

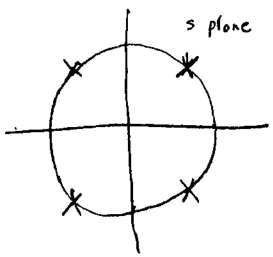
Pole locations for $N=2$

$$G(s)\,G(-s) = \frac{1}{1+(-1)^2 (s)^4} = \frac{1}{1+s^4}$$

poles are at $s^4 = -1$

$$s^4 = 1 \angle \pm\pi, \pm 3\pi, \ldots$$

$$s = 1 \angle \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \ldots$$



4 poles on unit circle

spacing $= \frac{\pi}{N}$

$$G(s) = \frac{1}{(s-(-.707+j.707))(s-(-.707-j.707))}$$

$$= \frac{1}{s^2+1.414s+1}$$

Notice:

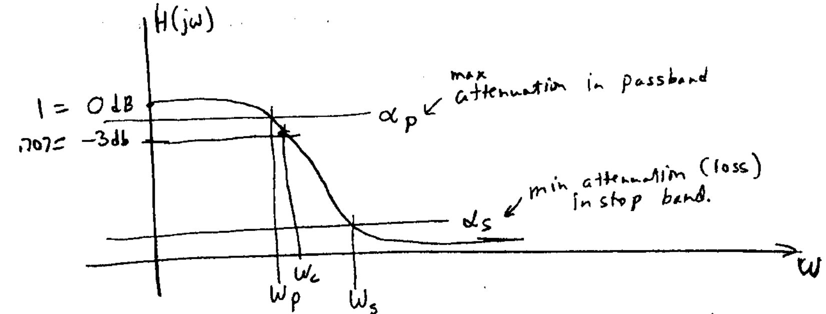
→ all poles lie on the (unit) circle.

→ We will later scale it for frequency. (so it is no longer a <u>unit</u> circle)

→ The $|H(s)|^2$ function has $2N$ poles, evenly spaced.

→ for $H(s)$, just take the ones (N of them) on the left half plane

Back to the general filter...



$$dB = 20 \log_{10} |H|$$

$$= 10 \log_{10} |H|^2$$

$|H| \geq -\alpha_p$ for $|\omega| \leq \omega_p$ (pass band)

$|H| \leq -\alpha_s$ for $|\omega| \geq \omega_s$ (stop band)

ω_c is -3dB frequency

by applying this, and rearranging the formula, we get:

(1):
$$N \geq \frac{\log_{10}\left[\left(10^{\alpha_s/10}-1\right)/\left(10^{\alpha_p/10}-1\right)\right]}{2\log_{10}\left[\omega_s/\omega_p\right]}$$

Use this formula to determine the required # of poles (N)

(2) To determine the 3dB frequency, given ω_p or ω_s ...

$$\omega_c = \frac{\omega_p}{\left[10^{\frac{\alpha_p}{10}}-1\right]^{\frac{1}{2N}}}$$

or...

$$\omega_c = \frac{\omega_s}{\left[10^{\frac{\alpha_s}{10}}-1\right]^{\frac{1}{2N}}}$$

(3) Find poles of $G(s)$ from tables.

Example:

Design a Butterworth Low pass filter with

$\alpha_p = .5$ dB for $|f| \leq f_p = 1$ KHz

$\alpha_s = 20$ dB for $|f| \geq f_s = 2$ KHz

step 1:

Find N (how many poles are needed)

$$N \geq \frac{\log_{10}\left[\left(10^{\overset{\alpha_s}{20/10}}-1\right)/\left(10^{\overset{\alpha_p}{.5/10}}-1\right)\right]}{2\log_{10}\left[\frac{2000 \leftarrow f_s}{1000 \leftarrow f_p}\right]}$$

$= 4.83$

use $N = 5$

step 2:

$$f_c = \frac{f_p}{\left[10^{\frac{\alpha_p}{10}}-1\right]^{\frac{1}{2N}}} = \frac{1000}{\left[10^{.5/10}-1\right]^{\frac{1}{10}}} = 1.234 \text{ kHz}$$

step 3:

$$G(s_n) = \frac{1}{(s_n+1)(s_n^2+1.618s_n+1)(s_n^2+.618s_n+1)}$$

s_n is s-normalized

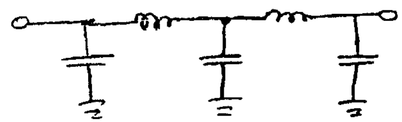
$$s_n = \frac{s}{\omega_c} = \frac{s}{2\pi(1234)} = \frac{s}{7753.5}$$

De-normalize:

$$G(s) = \frac{1}{\left(\frac{s}{7753}+1\right)\left(\left(\frac{s}{7753}\right)^2 + 1.618\frac{s}{7753}+1\right)\left(\left(\frac{s}{7753}\right)^2+.618\frac{s}{7753}+1\right)}$$

$$G(s) = \frac{(7753)^5}{(s+7753)\left(s^2+1.618(7753)s+(7753)^2\right)\left(s^2+.618(7753)s+(7753)^2\right)}$$

=



From this, you could determine the values for an analog filter, either R-L-C or "active" (with op-amps)

but we won't because this course covers Digital filters.

Near future topics!

How to transform to digital: ① Bilinear transformation
② Impulse invariance
③ Step invariance

Other types of filters: ① High pass
② Band pass

Other filter designs: ① Chebyshev
② Cauer

Homework - not to hand in.

Stanley: 6-4
6-9

Determine the transfer function for a Low-pass Butterworth filter with the following characteristics:

$\alpha_p = 1$ db for $|f| \le f_p = 15$ KHz

$\alpha_s = 40$ db for $|f| \le f_s = 19$ KHz

You will probably want to use the computer (Matlab, octave, whatever) to do the calculations.

HW – The big filter problem:

$\alpha_p = 1\,dB$ $\quad f_p = 15\ kHz$

$\alpha_s = 40\,dB$ $\quad f_s = 19\ kHz$

$$N \geq \frac{\log_{10}\left[\left(10^{\frac{\alpha_s}{10}}-1\right)/\left(10^{\frac{\alpha_p}{10}}-1\right)\right]}{2\log_{10}\left[f_s/f_p\right]}$$

$$= \frac{\log_{10}\left[\left(10^{\frac{40}{10}}-1\right)/\left(10^{\frac{1}{10}}-1\right)\right]}{2\log_{10}\left[19/15\right]}$$

$$= \frac{4.586}{.2} = 22.3 \rightarrow 23 \text{ poles}$$

(at least 11 op-amps; more likely 33)

New homework

Stanley: 6-5 $\quad$ 6-1

Repeat 19 kHz filter for Chebyshev

Chebyshev approximation

→ "Equal ripple" in pass band.

→ steeper roll-off in transition region. (better than any other "all pole")
(but ultimate rolloff is still 6N dB/octave or 20N dB/decade)

Idea:

Select points in the pass band, then find an interpolating polynomial through those points.

Points are selected by "Chebyshev polynomials".

Characteristics:

→ Oscillates between −1 and 1 for x between −1 and 1

→ Increases more rapidly for x > 1 than any other polynomial with these bounds.

Derive from either:

$$C_N(x) = \cos\left(N\cos^{-1}x\right) \quad (\text{use for } |x|<1)$$

or

$$C_N(x) = \cosh\left(N\cosh^{-1}x\right) \quad (\text{use for } |x|>1)$$

They don't look polynomial, but they are..
Consider a series expansion.

C_K can also be defined by a recursion expression:

$$C_{N+1}(x) = 2x\,C_N(x) - C_{N-1}(x)$$

$$C_0(x) = 1$$

$$C_1(x) = x$$

Applying the recursion expression:

$$C_2(x) = 2x^2 - 1$$

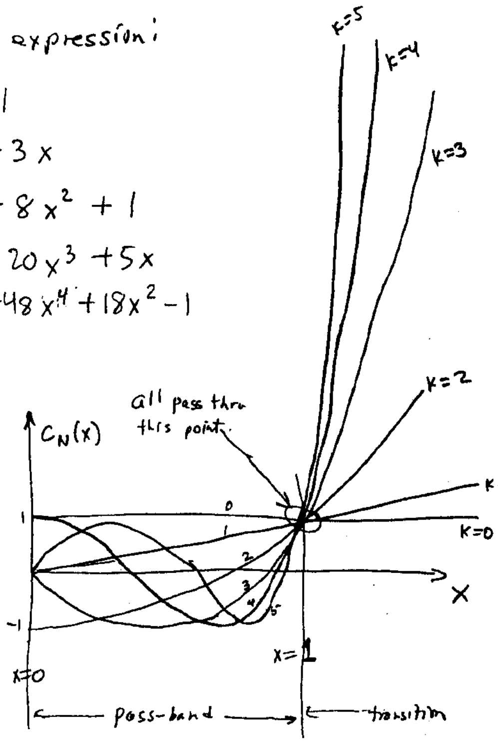
$$C_3(x) = 4x^3 - 3x$$

$$C_4(x) = 8x^4 - 8x^2 + 1$$

$$C_5(x) = 16x^5 - 20x^3 + 5x$$

$$C_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

etc.



The filter response: (without deriving it)

$$|H(j\omega)|^2 = \frac{A}{1+\epsilon^2 C_N^2(\omega/\omega_p)}$$

$$H(j\omega) = \frac{\sqrt{A}}{\sqrt{1+\epsilon^2 C_N^2(\omega/\omega_p)}}$$

N = order of polynomial = order of transfer function

ϵ^2 = parameter chosen for amount of ripple

α = chosen for proper DC value.

ω_p = pass band frequency (usually not 3dB)

For normalized, let $\alpha = 1$

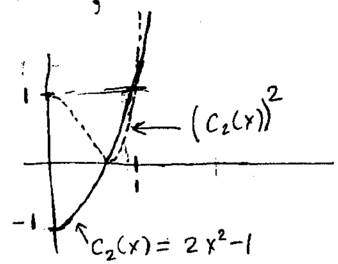
$$H(j\omega) = \frac{1}{\sqrt{1+\epsilon^2 C_N^2(\omega/\omega_p)}}$$

In pass band:

$$\frac{1}{\sqrt{1+\epsilon^2}} \le |H(j\omega)| \le 1, \quad 0 \le |f| \le f_p$$

Example! Let $N=2$

Then $C_2(x) = 2x^2 - 1$ ⟶



$(C_2(x))^2$ is the term we use in the formula

(now between 0 and 1)

ϵ^2 is a scale factor applied to $(C_2(x))^2$ (dotted line) to select amount of ripple.

In dB ...

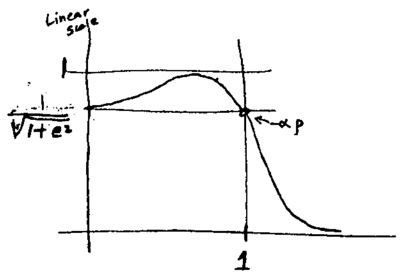
$$-\alpha_p = 20 \log \frac{1}{\sqrt{1+\epsilon^2}}$$

$$= -10 \log (1+\epsilon^2)$$

$$1+\epsilon^2 = 10^{\frac{\alpha_p}{10}}$$

$$\epsilon = \left[10^{\frac{\alpha_p}{10}} - 1\right]^{1/2}$$

α_p = max loss in passband (same def. as we used for Butterworth)



To find A ..

Set $\omega = 0$, $H(j\omega)$ = what you want it to be usually 1.

solve for A

To find the transfer function, substitute $\omega = \frac{s}{j}$

It is messy. I won't do it.

Use the tables.

Number of poles needed:

$$N \geq \frac{\cosh^{-1}\left[\left(10^{\frac{\alpha_s}{10}} - 1\right) / \left(10^{\frac{\alpha_p}{10}} - 1\right)\right]^{1/2}}{\cosh^{-1}\left[\omega_s / \omega_p\right]}$$

(by solving $H(j\omega)$ for N)

Comparison of Butterworth and Chebyshev.

Butterworth

$$|H(j\omega)| = \frac{1}{\sqrt{1+(\omega/\omega_c)^{2N}}} \Rightarrow \frac{1}{\left(\frac{\omega}{\omega_c}\right)^N}$$

is approximately equal to.

for large ω.

Chebyshev

$$|H(j\omega)| = \frac{1}{\sqrt{1 + C_N^2(\omega/\omega_c)}} \Rightarrow \frac{1}{C_N\left(\frac{\omega}{\omega_c}\right)}$$

for large ω

$X = C_N\left(\frac{\omega}{\omega_c}\right)$ is also an Nth order polynomial (see p.3 of today's notes)

For large X, the first term dominates (ignore the rest)

Large X (large ω) means well into the stop-band.

Example -- for 6th order filter

Butterworth ; $|H(j\omega)| \approx \frac{1}{\left(\frac{\omega}{\omega_c}\right)^6}$

Chebyshev : $|H(j\omega)| \approx \frac{1}{32\left(\frac{\omega}{\omega_c}\right)^6}$

$\rightarrow$ 32 times better attenuation in stop band.

Disadvantage is ripple.

Example: (7)

Design a Chebyshev low pass filter

$\alpha_p = .5\,\text{dB}$ for $|f| \le f_p = 1\,\text{kHz}$

$\alpha_s = 20\,\text{dB}$ for $|f| \ge f_s = 2\,\text{kHz}$

$$N \ge \frac{\cosh^{-1}\left[\left(10^{\frac{\overset{\alpha_s}{20}}{10}}-1\right)/\left(10^{\frac{\overset{\alpha_p}{.5}}{10}}-1\right)\right]^{1/2}}{\cosh^{-1}\left[\frac{\overset{f_s}{2000}}{\underset{f_p}{1000}}\right]}$$

$$= \frac{4.042}{1.317} = 3.06$$

use $N = 4$

From tables (.5 dB ripple)

$$H(s_n) = \frac{A}{(s_n^2 + .8467\,s_n + .3564)(s_n^2 + .3507\,s_n + 1.0635)}$$

To find numerator, set $s=0$, $H(s)=1$, solve.

$$1 = \frac{A}{(.3564)(1.0635)} \qquad A = .37903$$

Then de-normalize $\quad s_n = \frac{s}{\omega_p} = \frac{s}{2\pi(1000)} = \frac{s}{6283}$

$$H(s) = \frac{.37903}{\left(\left(\frac{s}{6283}\right)^2 + .8467\frac{s}{6283} + .3564\right)\left(\left(\frac{s}{6283}\right)^2 + .3507\frac{s}{6283} + 1.0635\right)}$$

$$= \frac{.37903\,(6283)^4}{(s^2 + .8467(6283)s + .3564(6283)^2)(s^2 + .3507(6283)s + 1.0635(6283)^2)}$$

Other approximation methods (8)

"inverted" Chebyshev (also known as "Chebyshev type II")

Ripples in stop band, flat in pass band.

Idea: if rejection is most important, move the zeros down to help get a steeper slope.

Also – "infinite" rejection at particular frequencies.

Very common in audio.

Can tune the rejection frequencies:

Example: FM stereo – put a notch (zero) at 19 kHz

"Cauer" or "Elliptic"

Ripples in both stop band and pass band.

Even steeper transition region.

"Bessel" or "maximally flat time delay"

Most linear phase.

Amplitude response is smooth